Stark-Wannier resonances and cubic exponential sums

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We consider the Schrödinger operator $H = -\frac{\partial^2}{\partial x^2} + v(x) - \epsilon x$ acting in $L^2(\mathbb{R})$. Here, v is an entire 1-periodic function, and ϵ is a positive constant. This operator is a model used to study the Bloch electron in a constant electric field. The parameter ϵ is proportional to the electric field. The spectrum of H is absolutely continuous and fills the real axis.

The operator attracted attention of both physicists and mathematicians after the discovery of the Stark-Wannier ladders. These are ϵ-periodic sequences of resonances, i.e., of the poles of the meromorphic extension of the resolvent from the upper half-plane of the complex plane across the spectrum, see [1, 3]. A series of papers was devoted to the asymptotics of the ladders as $\epsilon \to 0$, see, for example, [2]. The complexity of the problem is related to the fact that there are ladders exponentially close to the real axis. Actually, only the case of finite gap potentials v was understood relatively well: for these potentials, there is a finite number of ladders located near the real axis. It appeared that the ladders non-trivially "interact" as ϵ changes, and physicists conjectured that the behavior of the resonances strongly depends on the arithmetic nature of ϵ, see, e.g., [3].

We assume that $v(x) = 2\cos(2\pi x)$ and study the reflection coefficient $r(E)$ in the lower half-plane of the complex plane of the spectral parameter E. There, the function $E \mapsto 1/r(E)$ is an analytic ϵ-periodic function. Its zeros are resonances (and vice versa). Represent $1/r$ by its Fourier series, $1/r(E) = \sum_{m\in\mathbb{Z}} p(m)e^{2\pi i m E/\epsilon}$. Let $a(\epsilon) = \sqrt{2/\epsilon}\,\pi e^{i\pi/4}$. We prove that, as $m \to \infty$,

$$p(m) = a(\epsilon)\sqrt{m}\, e^{-2\pi i\omega m^3 - 2m\log(2\pi m/e) + O(\log^2 m/m)}, \quad \omega = \left\{\frac{\pi^2}{3\epsilon}\right\}, \tag{1}$$

where $\{\cdot\}$ is the fractional part, and the error term estimate is locally uniform in $\epsilon > 0$.

Obviously, the asymptotic behavior of $1/r(E)$ as $\operatorname{Im} E \to -\infty$, is determined by the Fourier series terms with large positive m, and so, roughly, as $\operatorname{Im} E \to -\infty$,

$$\frac{1}{r(E)} \approx a(\epsilon)\,\mathcal{P}(E/\epsilon), \quad \mathcal{P}(s) = \sum_{m\geq 1} \sqrt{m}\, e^{-2\pi i\omega m^3 - 2m\log(2\pi m/e) + 2\pi i m s}. \tag{2}$$

It is worth to compare the function $1/r$ with the cubic exponential sums $\sum_{n=1}^{N} e^{-2\pi i\omega n^3}$. They were extensively studied for large N in the analytic number theory, see [4], and proved to depend strongly on the arithmetic nature of ω. For $1/r$, this appears to be true too. In the talk, for $\omega \in \mathbb{Q}$, we describe in detail behavior of the resonances far from the real axis. In particular, we show that, if $\omega = p/q$ with coprime integers $0 < p < q$, then their asymptotics are determined by beautiful and nontrivial properties of the complete rational exponential sums $\sum_{l=0}^{q-1} e^{-2\pi i\,\frac{pl^3 - ml}{q}}$, $m \in \mathbb{Z}$.

The talk is based on a joint work with Frederic Klopp.

References

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