Khovanov skein homology for links in the thickened torus

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joint work with Boyu Zhang

Jones polynomial

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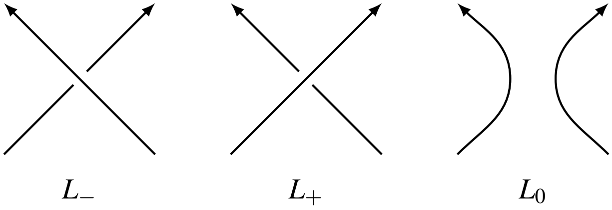
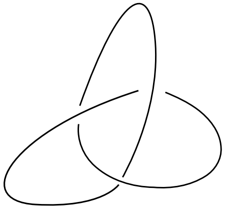


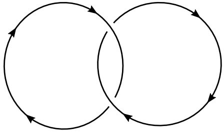
Definition

- $V(\text{unknot}) := 1$;
- $t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$.

Examples



(a) The trefoil knot.



(b) The positive Hopf link.

- $V(\text{trefoil}) = t + t^3 - t^4$;
- $V(\text{positive Hopf link}) = -(t^{1/2} + t^{5/2})$,
  $V(\text{negative Hopf link}) = -(t^{-1/2} + t^{-5/2})$.

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Still open. No counter-example among knots up to 24 crossings (Tuzun and Sikora, 2020).

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Replace numerical invariants with category-theoretic invariants (e.g. functors).

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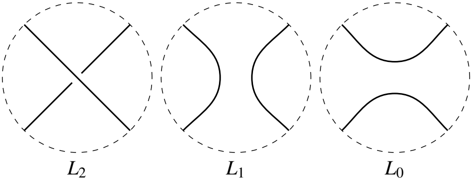
- Alexander polynomial $\rightsquigarrow$ (Heegaard, instanton, monopole) knot Floer homology;
- Jones polynomial $\rightsquigarrow$ Khovanov homology.

Kauffman bracket



Definition

Let U be the unknot.

$$\langle U \rangle = 1$$
$$\langle U \sqcup L \rangle = (q + q^{-1}) \langle L \rangle$$
$$\langle L_2 \rangle = \langle L_0 \rangle - q \langle L_1 \rangle$$

Kauffman's definition of the Jones polynomial

Given a link L, let $J(L)(q) = V(L)_{-t^{1/2}=q} \in \mathbb{Z}[q, q^{-1}]$. Then

$$J(L)(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$$

where n_+ and n_- denote the numbers of positive and negative crossings of the diagram of L.

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If we replace the first equation with $\langle U \rangle = q + q^{-1}$ then we obtain the unnormalized Jones polynomial $\hat{J}(L) = (q + q^{-1})J(L)$.

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Khovanov homology

- In order to define the Khovanov homology, we need to categorify the definition of the Kauffman bracket.
- Let $V := \mathbb{Z}\{v_+, v_-\}$ with $\mathrm{qdeg}\, v_- = -1, \mathrm{qdeg}\, v_+ = 1$.
- Define

$$\begin{aligned} [\![U]\!] &= 0 \to V \to 0 \\ [\![U \sqcup L]\!] &= V \otimes [\![L]\!] \\ [\![L_2]\!] &= \text{Cone}([\![L_0]\!] \xrightarrow{d} [\![L_1]\!]\{1\}) \end{aligned}$$

where $\{1\}$ means we increase the q-grading of the chain complex by 1.

Khovanov homology

- Suppose the diagram D of L has N crossings. Given any $c \in \{0,1\}^N$, we could smooth all the crossings of D to obtain a collection of circles D_c in $\mathbb{R}^2$;

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- Assign V to each circle in D_c and let $V(D_c)$ be the tensor product of these copies of V;
- For each vertex of the N-dimensional cube, we have an abelian group $V(D_c)$. For each edge of the cube, we want to define a map $d_{cc'} : V(D_c) \to V(D_{c'})$ where c and c' differs at only one crossing;

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- In this way, we obtain a chain complex

$$(C(D), d) = (\bigoplus_c V(D_c), \sum_{c,c'} d_{cc'})$$

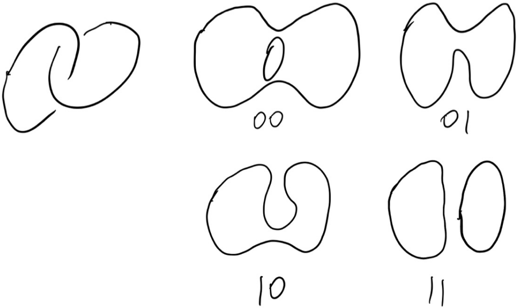

Figure: The 4 smoothings of the Hopf link.

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\nabla} & V \\ \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \Delta} \\ V & \xrightarrow{-\Delta} & V \otimes V \end{array}$$

The multiplication and comultiplication on V

$$\nabla : V \otimes V \to V$$
$$v_+ \otimes v_\pm \mapsto v_\pm$$
$$v_\pm \otimes v_+ \mapsto v_\pm$$
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- Define $\mathrm{Kh}(L) := H([\![L]\!][-n_-]\{n_+ - 2n_-\})$. We use $\mathrm{Kh}^{i,j}(L)$ to denote the summand with h-degree i and q-degree j.

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Theorem (Khovanov, 2000)

The homology group $\mathrm{Kh}^{i,j}(L)$ does not depend on the choice of the diagram of L.

Khovanov homology

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$$\chi_q(\mathrm{Kh}(L)) := \sum_{i,j} (-1)^i q^j \operatorname{rank} \mathrm{Kh}^{i,j}(L) = \hat{J}(L).$$

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Examples

- $\mathrm{Kh}($n component unlink$) = (\mathbb{Z}_{(0,1)} \oplus \mathbb{Z}_{(0,-1)})^{\otimes n}$.
- $\mathrm{Khr}($positive Hopf link$) = \mathbb{Z}_{(0,1)} \oplus \mathbb{Z}_{(2,5)}$,
  $\mathrm{Kh}($positive Hopf link$) = \mathbb{Z}_{(0,0)} \oplus \mathbb{Z}_{(0,2)} \oplus \mathbb{Z}_{(2,4)} \oplus \mathbb{Z}_{(2,6)}$.
- $\mathrm{Khr}($trefoil$) = \mathbb{Z}^3$, $\mathrm{Kh}($trefoil$) = \mathbb{Z}^4 \oplus \mathbb{Z}/2$.

- Khovanov homology detects the unknot (Kronheimer-Mrowka, 2011), trefoil knot (Baldwin-Sivek, 2018), figure eight knot (Baldwin-Dowlin-Levine-Lidman-Sazdanovic, 2020), the torus knot $T_{2,5}$ (Baldwin-Hu-Sivek, 2021);

Properties and Applications

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- Conway knot is not slice (Piccirillo, 2018);
- Possibly verify potential counter-examples of 4-dimensional Poincaré conjecture (Freedman-Gompf-Morrison-Walker, Manolescu-Piccirillo).

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Asaeda-Przytycki-Sikora's generalization

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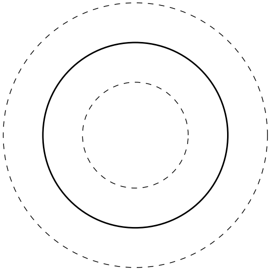
- Asaeda, Przytycki and Sikora generalized Khovanvov's definition to I-bundles over compact surfaces (possibly with boundary);
- We will focus on $(-1,1) \times \Sigma$ where Σ is an orientable compact surface.

- Consider links in $(-1,1) \times A$ where A is an annulus. We can resolve crossings of L to obtain a collection of circles in A.

Annular Khovanov homology

- Consider links in $(-1,1)\times A$ where A is an annulus. We can resolve crossings of L to obtain a collection of circles in A.
- Two type of circles: trivial circles (that bound disks) and homologically non-trivial circles;



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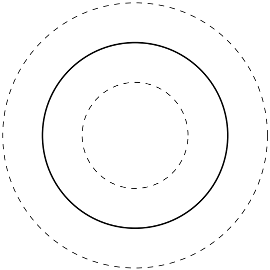

- Assign $V=\mathbb{Z}\{v_+,v_-\}$ to a trivial circle and $W=\mathbb{Z}\{w_+,w_-\}$ to a non-trivial circles ($\operatorname{qdeg} w_\pm=\pm1$).

Differentials

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- A trivial circle and a non-trivial circle merge into a non-trivial circle (or the other way around):

$$V \otimes W \to W \qquad W \to V \otimes W$$

$$v_+ \otimes w_\pm \mapsto w_\pm \qquad w_+ \mapsto w_+ \otimes v_-$$

$$v_- \otimes w_\pm \mapsto 0 \qquad w_- \mapsto w_- \otimes v_-$$

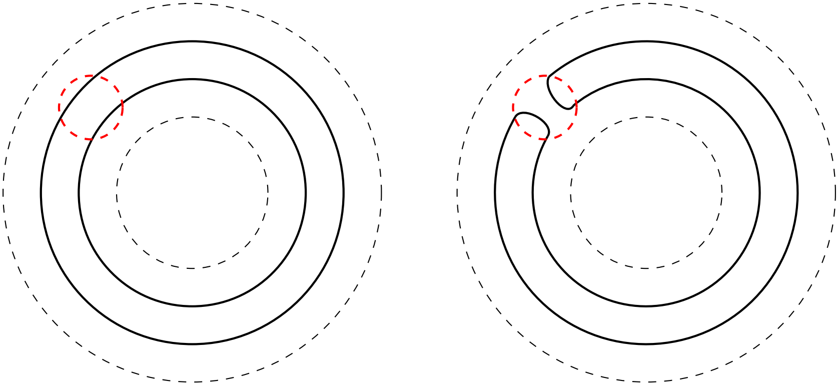

Figure: Two non-trivial circles merge into a trivial circle in an annulus

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- The annular Khovanov homology (AKh) is triply graded: h-grading, q-grading, f-grading.

Torus case

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Torus case

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- The homology $\mathrm{TKh}(L)$ for a link L in $(-1,1) \times T^2$ is $\mathbb{Z}^2 \oplus \mathbb{Z}\mathcal{C}$-graded.

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Detection properties

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- AKh detects the unlink and braid closures in the thickened annulus;
- TKh detects the unlink and torus links in the thickened torus;
- Given $[c] \in \mathcal{C}$, the $\mathbb{Z}\mathcal{C}$-grading of $\mathrm{TKh}(L)$ is supported in $\mathbb{Z}\{[c]\}$ if and only if L is disjoint from $(-1,1) \times c$ after isotopy.

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Theorem

A connected surface $\Sigma \subset Y$ is norm-minimizing if and only if

$$E(\mathrm{I}(Y, L), \mu(\Sigma), 2g(\Sigma) - 2 + |\Sigma \cap L|) \neq 0$$

.

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- The $\mathbb{Z}\mathcal{C}$-grading on $\mathrm{TKh}(L)$ descends to a $\mathbb{Z}$-grading under the map $[\gamma] \mapsto \gamma \cdot c$.

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Proposition

There is a spectral sequence from $\mathrm{TKh}(L)$ to $\mathrm{THI}(L)$ which preserves the c-grading.

Sketch of the proof of the last part of the main theorem

- Given $L \subset (-1,1) \times T^2$, suppose the $\mathbb{Z}\mathcal{C}$ grading of $\mathrm{TKh}(L)$ is supported at $\mathbb{Z}\{[c]\}$;

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- Using the Thurston norm detection property of instanton Floer homology, we could find a norm 0 surface Σ whose homology class is $[S^1 \times c]$. In particular, Σ is a torus.
- Σ can be isotoped to $S^1 \times c$.

Thanks!