Quantum information related physics and mathematics

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Superposition principle
quantum mechanics ⟹

- Quantum coherence
- Uncertainty relations
- Wave-particle duality
- Quantum correlations

  discord, entanglement, nonlocality, …

Quantum state

Quantum measurement

Quantum states

Pure state: vector $\mathcal{H} \otimes \mathcal{H} \otimes ... \otimes \mathcal{H}$

$$|\psi\rangle = \sum_{i,j,\ldots,k=1}^{N} a_{ij\ldots k}|ij\ldots k\rangle, \qquad a_{ij\ldots k} \in \mathbb{C}$$

$$\sum_{i,j,\ldots,k=1}^{N} a_{ij\ldots k} a^{*}_{ij\ldots k} = 1 \qquad |ij\ldots k\rangle \equiv |i\rangle \otimes |j\rangle \otimes \ldots \otimes |k\rangle$$

Mixed state: density matrix

Density matrix: Hermitian, Semipositive $Tr(\rho) = 1$

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|, \qquad 0 < p_i \leq 1, \qquad \sum p_i = 1$$

$$\langle\psi_i| = (|\psi_i\rangle)^{\dagger}$$

Quantum measurement

Observable self-adjoint operators

$$O = \sum_i \lambda_i |v_i\rangle\langle v_i|$$

real eigenvalues λ_i

eigenvectors $|\nu_i\rangle$

$\{|\nu_i\rangle\langle\nu_i|\}$ probability $p_i = Tr(|\nu_i\rangle\langle\nu_i|\,\rho)$

Mean value $\bar{O} = Tr(O\,\rho)$

von Neumann measurement

$$M = \{|a_n\rangle\langle a_n|\} \qquad \sum |a_n\rangle\langle a_n| = I$$

POVM (Positive operator-valued measure)

$$M = \{M_\mu\}, \quad M_\mu \geq 0, \quad \sum M_\mu = I$$

Evolution

$$\rho \Longrightarrow U\rho U^\dagger$$

Quantum channel

$$\rho \Longrightarrow \sum_i E_i \rho E_i^\dagger$$

E_i Kraus operators $\sum_i E_i^\dagger E_i = I$

Quantum coherence

Incoherent states

$$\sigma = \sum_{i=1}^{d} p_i |i\rangle\langle i|$$

Measures of coherence

l_1 - norm coherence

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|$$

Relative entropy of coherence

$$C^r(\rho) = S(\rho_d) - S(\rho)$$

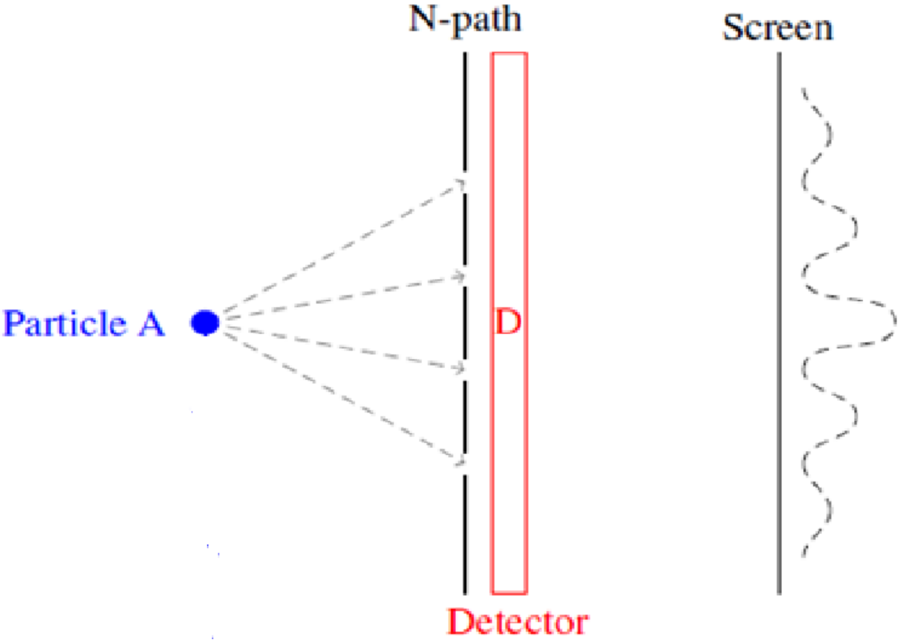
$$S(\rho) = -\mathrm{Tr}\left[\rho \log \rho\right]$$ von Neumann entropy

Generalized Multipath Wave-particle Duality in a Delayed-choice Experiment

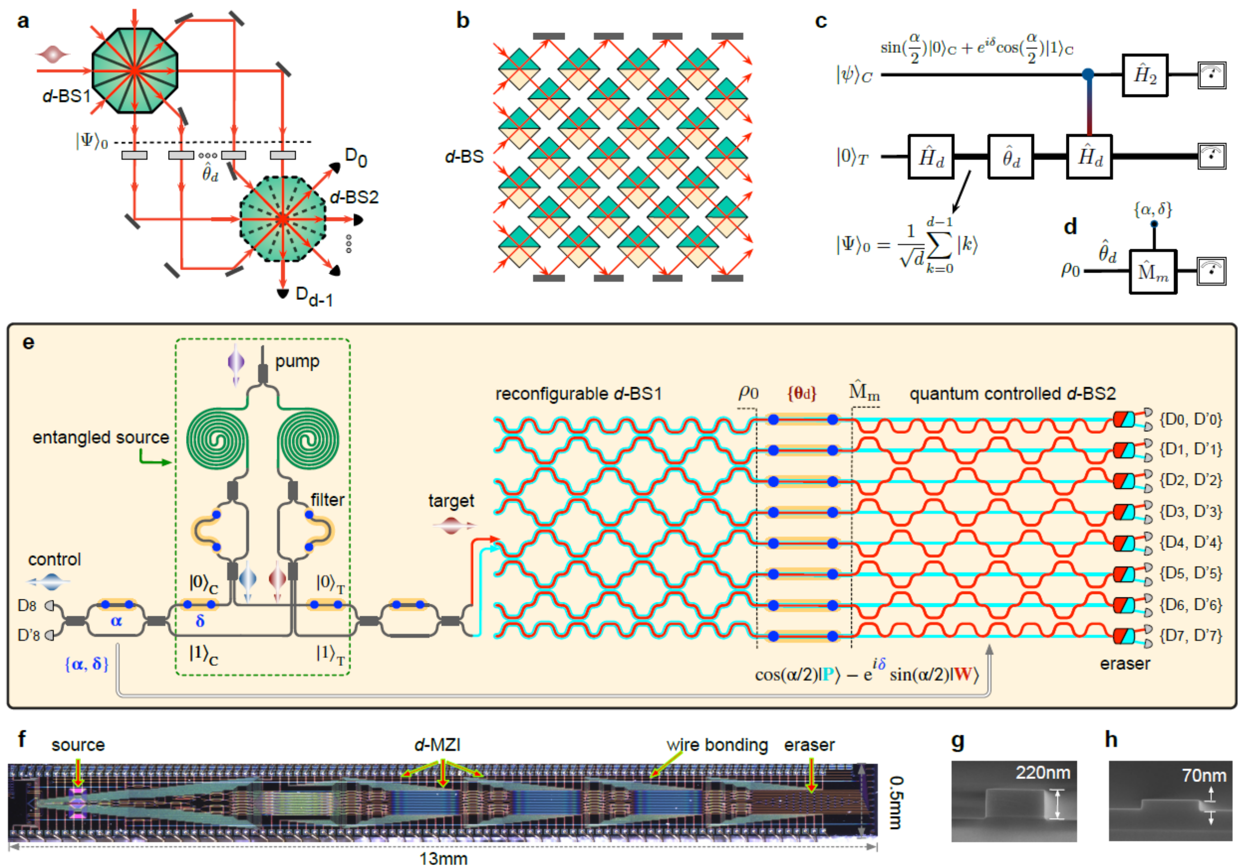
$$\mathcal{C}_d = \frac{1}{d-1}\sum_{i\neq j}\left|\rho_{ij}\right|,$$

$$\mathcal{D}_d = \sqrt{1-\left(\frac{1}{d-1}\sum_{i\neq j}\sqrt{\rho_{ii}\rho_{jj}}\right)^2},$$

$$\mathcal{C}_d^2 + \mathcal{D}_d^2 \leqslant 1,$$



Quantum delayed-choice multipath experiment on a large-scale silicon-integrated quantum nanophotonic chip



X. Chen, Y. Deng, S. Liu, T. Pramanik, J. Mao, J. Bao, C. Zhai, H. Yuan, J. Guo, S.M. Fei, B. Tang, Y. Yang, Z. Li, Q. He, Q. Gong, J. Wang, Nature Commun. 12 (2021) 2712

Robustness of coherence

$$\mathcal{C}(\rho) = \min_{\tau \in \mathcal{D}} \left\{ s \geq 0 \middle| \frac{\rho + s\tau}{1+s} = \delta \in \mathcal{S} \right\}$$

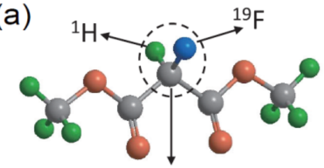
Phys. Rev. Lett. 116, 150502 (2016)

Phase discrimination

$$\mathcal{A}(\rho) \equiv \frac{p_{succ}(\rho)}{p_{succ}(\delta)} = 1 + \mathcal{C}(\rho)$$

NMR

(a)



	^{13}C	^{1}H	^{19}F	T_2	T_1	T_2^*
^{13}C	8558 Hz			1.1 s	2.9 s	152 ms
^{1}H	160.6 Hz	2035 Hz		1.2 s	2.8 s	136 ms
^{19}F	-194.4 Hz	47.9 Hz	-73846 Hz	1.3 s	3.1 s	163 ms

W.Q. Zheng, Z.H. Ma, H.Y. Wang, S.M. Fei, X.H. Peng, Phys. Rev. Lett. 120 (2018) 230504

Max- relative entropy of coherence

$$D_{\max}(\rho||\sigma) := \min\{\lambda : \rho \leq 2^{\lambda}\sigma\}$$

$$C_{\max}(\rho) := \min_{\sigma \in \mathscr{I}} D_{\max}(\rho||\sigma)$$

Subchannel discrmination

K.F. Bu, U. Singh, S.M. Fei, A.K. Pati, J.D. Wu,
Phys. Rev. Lett. 119 (2017) 150405

Coherence measures with respect to general quantum measurements

Incoherent $\langle i|\rho|j\rangle = 0$ ⇨

$$|i\rangle\langle i|\rho|j\rangle\langle j| = 0$$

POVM $E = \{E_i\}_{i=1}^{n}$ $E_i \rho E_j = 0, \quad \forall i \neq j$

Let $E = \{E_i = A_i^{\dagger} A_i\}_{i=1}^{n}$ be a POVM

l_1 norm of coherence

$$C_{l_1}(\rho, E) = \sum_{i \neq j} \|A_i \rho A_j^{\dagger}\|_{\mathrm{tr}}$$

J.W. Xu, L.H. Shao, S.M. Fei, Phys. Rev. A 102 (2020) 012411

Maximum Relative Entropy of Coherence for Quantum Channels

$$C_{\max}(\phi) = \min_{\widetilde{\phi} \in \mathcal{IC}_{AB}} D_{\max}(\phi | \widetilde{\phi})$$

where $\mathcal{IC}_{AB}$ is the set of incoherent channels

$$D_{\max}\left(\phi | \widetilde{\phi}\right) = \min\left\{\lambda : J_{\phi} \leq 2^{\lambda} J_{\widetilde{\phi}}\right\}$$

Choi matrix

Z.X. Jin, L.M. Yang, S.M. Fei, X. Li-Jost, Z.X. Wang, G.L. Long, C.F. Qiao, Sci. China Phys. Mech. & Astron. 64 (2021) 280311

Quantum entanglement

Mixed state: density matrix

Density matrix: Hermitian, Semipositive $Tr(\rho) = 1$

n-partite system

$$\rho = \sum \mathrm{p_i} \rho_1^{\mathrm{i}} \otimes \rho_2^{\mathrm{i}} \otimes \ldots \otimes \rho_{\mathrm{n}}^{\mathrm{i}}$$

fully separable

Measure of quantum entanglement

Parametrized entanglement measure

q-concurrence

$$C_q(|\psi\rangle_{AB}) = 1 - \mathrm{Tr}\rho_A^q \qquad q \geqslant 2 \qquad \rho_A = Tr_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$$

$$C_q(\rho_{AB}) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_q(|\psi_i\rangle_{AB}) \qquad \rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|_{AB}$$

$$C_q(\rho) \geqslant \frac{\left[\max\left\{\|\rho^{T_A}\|_1^{q-1}, \|\mathcal{R}(\rho)\|_1^{q-1}\right\} - 1\right]^2}{m^{2q-2} - m^{q-1}}$$

X. Yang, M.X. Luo, Y.H. Yang, S.M. Fei, Phys. Rev. A 103 (2021) 052423

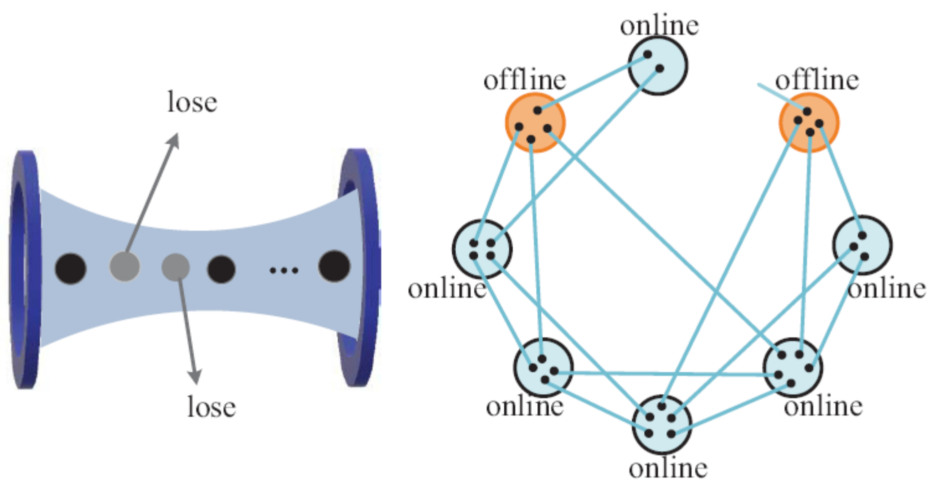
PHYSICAL REVIEW RESEARCH **3**, 043120 (2021)

Robust multipartite entanglement without entanglement breaking

Ming-Xing Luo[1,*] and Shao-Ming Fei[2,3]

[1] *School of Information Science and Technology, Southwest Jiaotong University, Chengdu 610031, China*

[2] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*



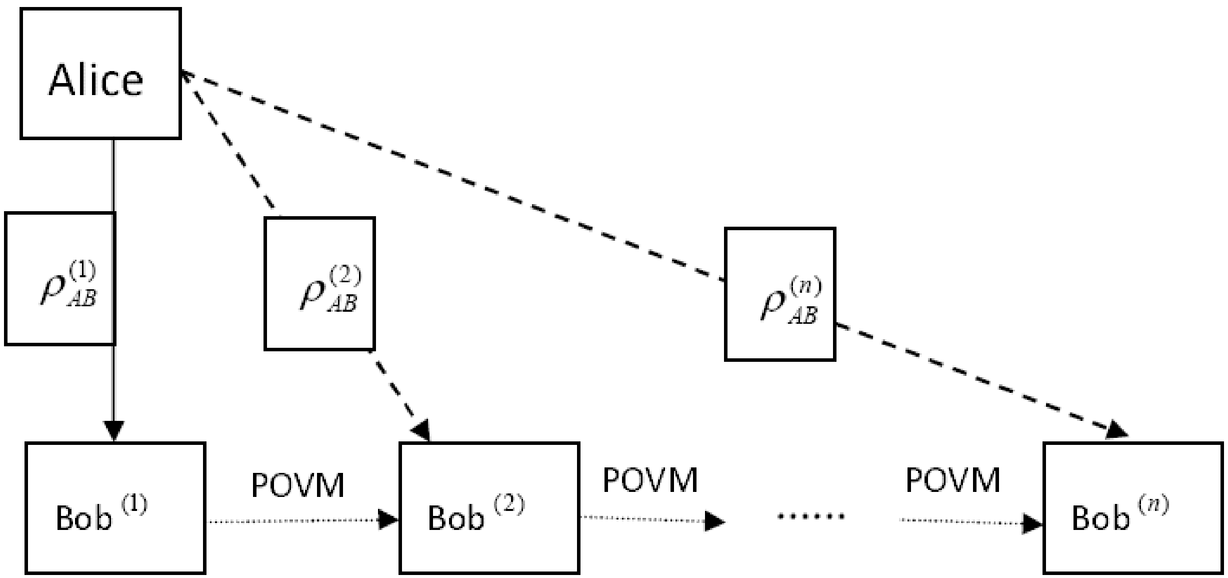
Blindly verifying partially unknown entanglement

M.X. Luo, S.M. Fei, J.L. Chen, iScience 5 (2022) 103972

Sharing quantum nonlocality with independent observables

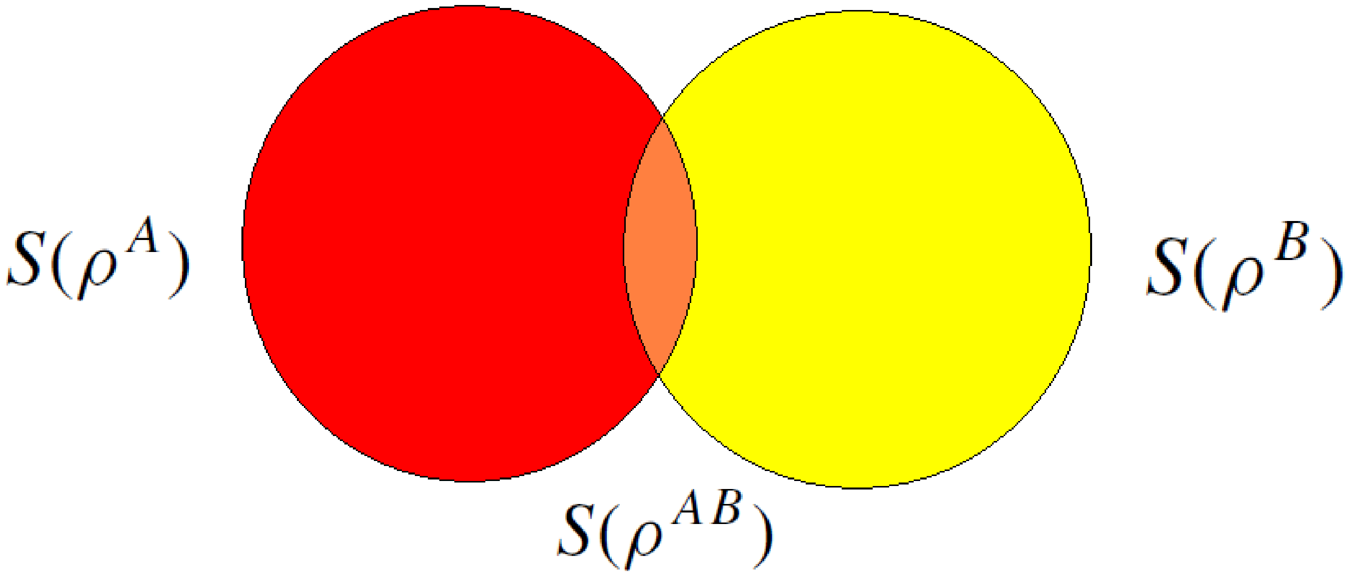
$$\mathcal{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$$

$$|\langle \mathcal{B} \rangle_\rho| \leq 2 \qquad \langle \mathcal{B} \rangle = tr(\rho \mathcal{B})$$



T.G. Zhang, S.M. Fei, Phys. Rev. A 103 (2021) 032216

Quantum discord

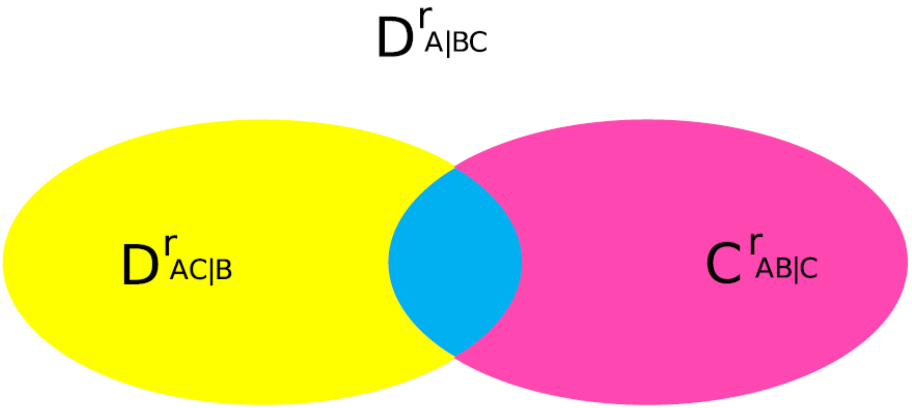


C. Radhakrishnan, M. Laurière, and T. Byrnes, Phys. Rev. Lett. **124**, 110401 (2020).

Quantum Discord for Multi-qubit Systems

B. Li, C.L. Zhu, X.B. Liang, B.L. Ye, S.M. Fei, Phys. Rev. A 104 (2021) 012428

Quantum coherence bounds the distributed discords



$$C^r_{A|B}(\rho_{AB}) = \min_{\sigma_{AB}\in\mathcal{I}_{A|B}} S(\rho_{AB}||\sigma_{A|B})$$

B-incoherent states $\quad \sigma_{AB} = \sum_i p_i \sigma_A^i \otimes |i\rangle_B\langle i|.$

$$D^r_{A|BC}(\rho) - D^r_{AC|B}(\rho) \leq C^r_{AB|C}(\rho)$$

Z.X. Jin, X. Li-Jost, S.M. Fei, C.F. Qiao, NPJ Quant. Inform. 8 (2022) 33

Quantum information masking:

$$\mathcal{U}: \quad |a_s\rangle_A \in \Omega \subseteq \mathcal{H}_A$$

$$\Longrightarrow \quad |\Psi_s\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho_A = \mathrm{Tr}_B(|\Psi_s\rangle_{AB}\langle\Psi_s|)$$

$$\rho_B = \mathrm{Tr}_A(|\Psi_s\rangle_{AB}\langle\Psi_s|) \text{ for all } s$$

K. Modi, A. K. Pati, A. Sen, U. Sen, Phys. Rev. Lett. 120, 230501 (2018)

B. Li, S.H Jiang, X.B. Liang, X. Li-Jost, H. Fan, S.M. Fei, Phys. Rev. A 99 (2019) 052343

Deterministic versus probabilistic

Complete Characterization of Qubit Masking

$$\cos\frac{x}{2}|0\rangle + e^{iy}\sin\frac{x}{2}|1\rangle \equiv |(x,y)\rangle$$

$$x \in [0,\pi] \qquad y \in [0,2\pi)$$

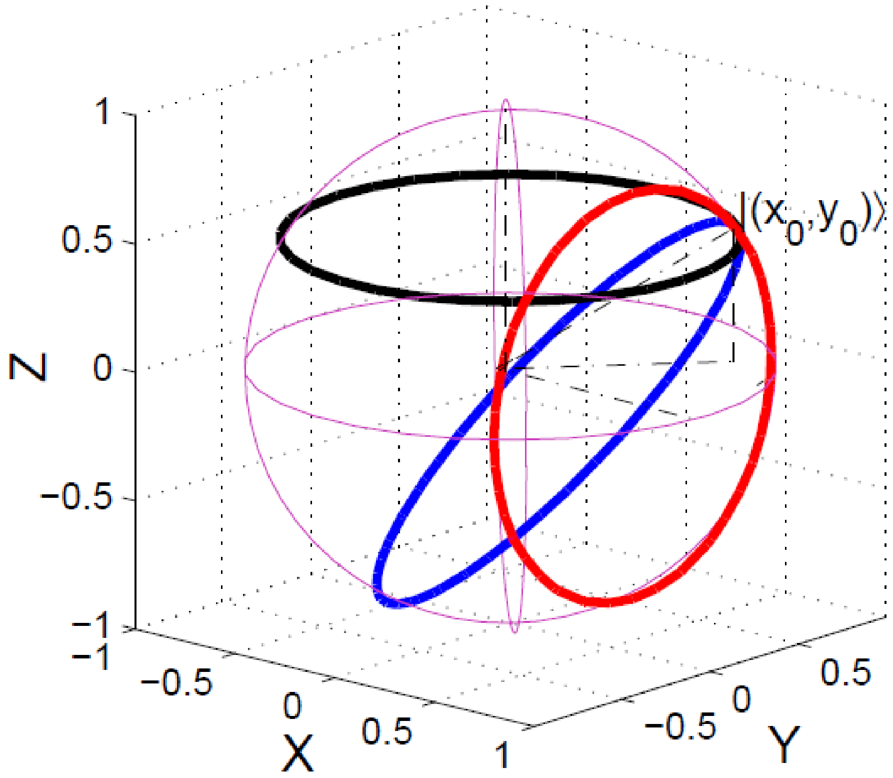
area measure of the point set $[0,\pi] \times [0,2\pi]$

Theorem 2.—No nonzero linear operator can mask the nonzero measure set of qubit states.

$$\cos x = Z \qquad \sin x \cos y = X \qquad \sin x \sin y = Y$$

$$X^2 + Y^2 + Z^2 = 1$$

Quantum secret sharing



X.B. Liang, B. Li, S.M. Fei, Phys. Rev. A 100 (2019) 030304 (R)

High dimensional case

X.B. Liang, B. Li, S.M. Fei, H. Fan, Phys. Rev. A, 101 (2020) 042321

Tensor network compressed sensing and machine learning

the nth pixel $(0 \leqslant x_n \leqslant 1)$

$$x_n \to |s(x_n)\rangle = \cos(x_n\pi/2)|0\rangle + \sin(x_n\pi/2)|1\rangle$$

"3" images $\Longrightarrow$ $|\Psi\rangle$

$$|\Phi\rangle = \prod_{x_n \in \{x^{[\mathrm{sent}]}\}} \langle s(x_n)|\Psi\rangle / C$$

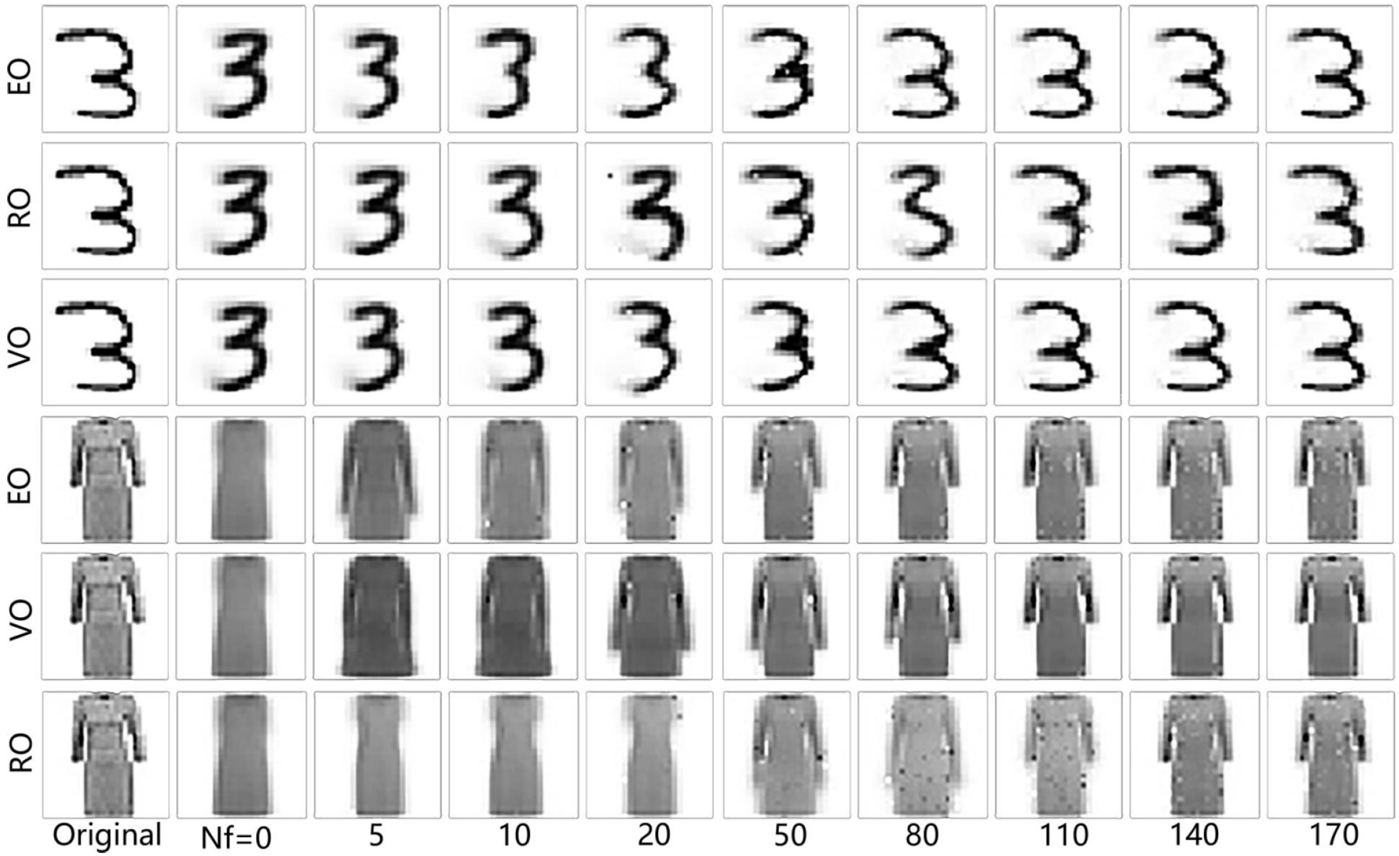


FIG. 7. Examples of the original and generated images in MNIST and fashion-MNIST in the entanglement order (EO), random order (RO), and variance order (VO). The number of known features N_f varies from 0 to 170, while the total number of features in an image is 784. We take the bond dimension of the generative MPS as $\chi = 40$.

S.J. Ran, Z.Z. Sun, S.M. Fei, G. Su, M. Lewenstein, Phys. Rev. Research 2 (2020) 033293

Quantum uncertainty relations

POVM (Positive operator-valued measure)

$$M = \{M_\mu\}, \quad M_\mu \geq 0, \quad \sum M_\mu = I$$

Joint measurability

$$M_1 = \{M^1_\mu\}, \quad M^1_\mu \geq 0, \quad \sum M^1_\mu = I$$

$$M_2 = \{M^2_\nu\}, \quad M^2_\nu \geq 0, \quad \sum M^2_\nu = I$$

$$M_{12} = \{M^{12}_{\mu\nu}\}, \quad M^{12}_{\mu\nu} \geq 0, \quad \sum M^{12}_{\mu\nu} = I$$

$$M^1_\mu = \sum_\nu M^{12}_{\mu\nu}, \quad M^2_\nu = \sum_\mu M^{12}_{\mu\nu}$$

jointly measurable

Qubit case:

$$A_{\pm} = (I \pm \mathbf{a} \cdot \boldsymbol{\sigma})/2$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

Pauli matrices

$$B_{\pm} = (I \pm \mathbf{b} \cdot \boldsymbol{\sigma})/2$$

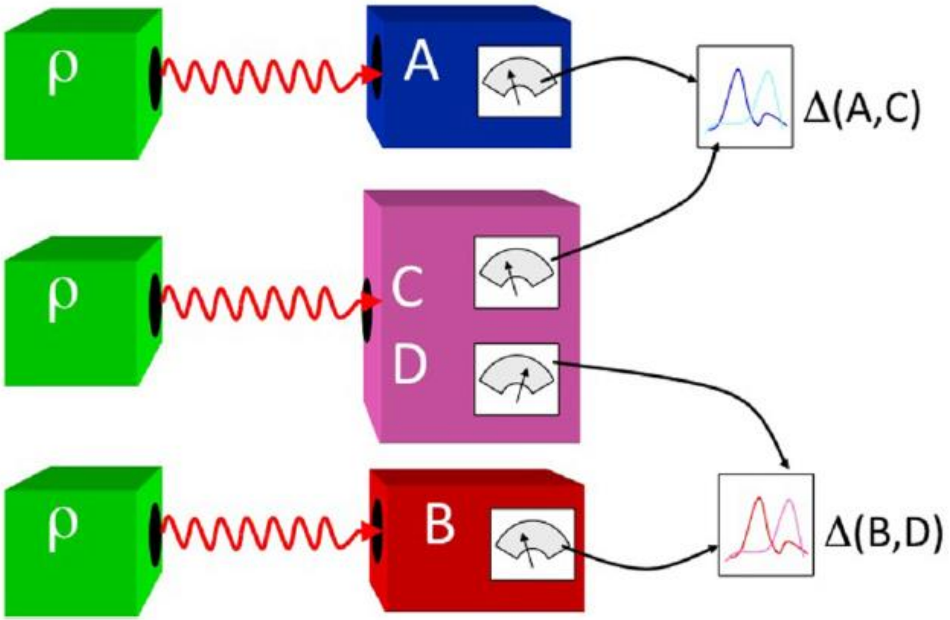
$$|\boldsymbol{a}| \leqslant 1$$

Not jointly measurable: $|\boldsymbol{a} + \boldsymbol{b}| + |\boldsymbol{a} - \boldsymbol{b}| > 2$

Probability: $\mathbf{Pr}\left[\Theta_{2,2}^{\mathrm{NJM}}(0,0)\right] = \frac{3}{5}$

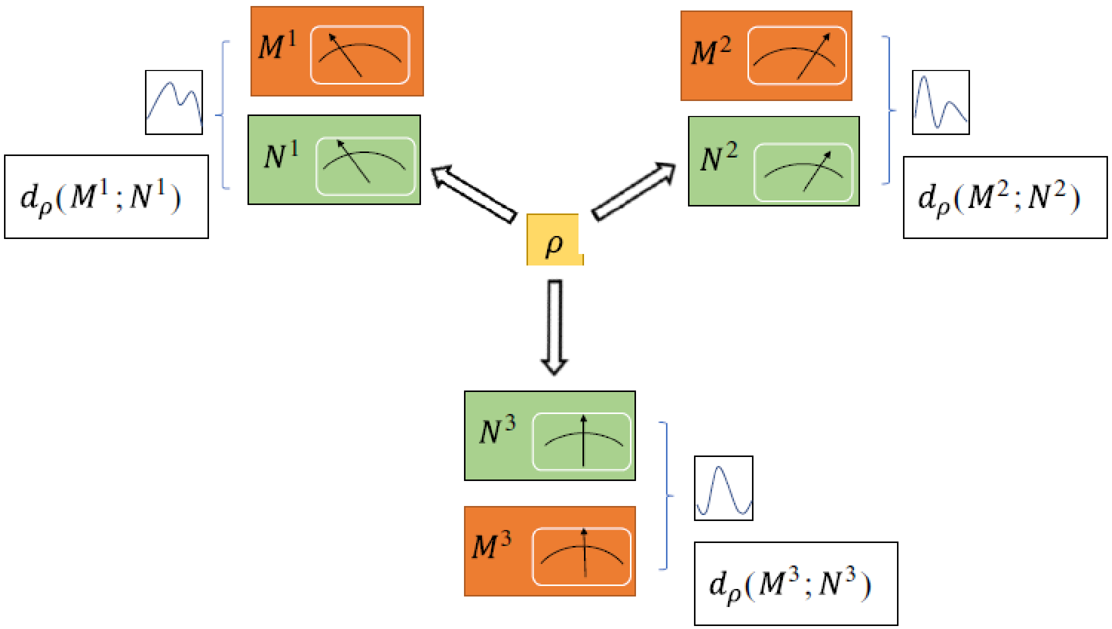
L. Zhang, H. Xiang, X. Li-Jost, S.M. Fei,
Phys. Rev. E 100, 062139 (2019)

Heisenberg's measurement uncertainty relation based on statistical distances



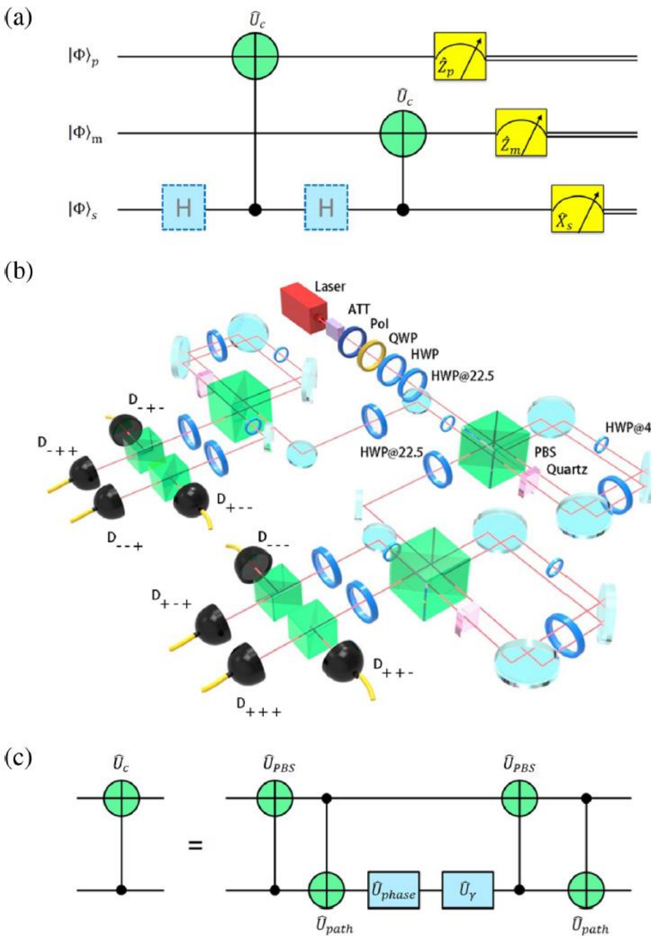
W.C. Ma, Z.H. Ma, H.Y. Wang, Y. Liu, Z.H. Chen, F. Kong, Z.K. Li, M.J. Shi, F.Z. Shi, S.M. Fei, J.F. Du, Phys. Rev. Lett. 116 (2016) 160405

Uncertainties of genuinely incompatible triple measurements



H.H. Qin, T.G. Zhang, L. Jost, C.P. Sun, X. Li-Jost, S.M. Fei, Phys. Rev. A 99 (2019) 032107

Error-Disturbance Trade-off in Sequential Quantum Measurements



$$\epsilon(\hat{A}) + \eta(\hat{B}) \geq \xi_{G,\max}$$

Y.L. Mao, Z.H. Ma, R.B. Jin, Q.C. Sun, S.M. Fei, Q. Zhang, J. Fan, J.W. Pan,
Phys. Rev. Lett. 122 (2019) 090404

Uncertainty Equality with Quantum Memory

$$\rho_{AB} \qquad \{|i_\theta\rangle \,|\, i = 1, 2, \cdots, d\}$$

$$\rho_{\theta B} = \sum_{i=1}^{d} |i_\theta\rangle_A \langle i_\theta| \otimes_A \langle i_\theta| \rho_{AB} |i_\theta\rangle_A$$

Conditional linear entropy $S_L(\rho) = 1 - \mathrm{Tr}(\rho^2)$

$$S_L(\theta|B) := S_L(\rho_{\theta B}) - S_L(\rho_B) = \mathrm{Tr}(\rho_B^2) - \mathrm{Tr}(\rho_{\theta B}^2)$$

Mutually unbiased bases (MUB)

$$|\langle b_i | c_j \rangle| = \frac{1}{\sqrt{d}}, \quad \forall\, i, j = 1, 2, \ldots, d$$

A complete set of d + 1 MUBs

$$\sum_{\theta=1}^{d+1} S_L(\theta|B) = d\left(\mathrm{Tr}(\rho_B^2) - \frac{1}{d}\mathrm{Tr}(\rho_{AB}^2)\right)$$

$$S_L(x|B) + S_L(y|B) + S_L(z|B) = 2\mathrm{Tr}(\rho_B^2) - \mathrm{Tr}(\rho_{AB}^2)$$

H. Wang, Z. Ma, S. Wu, W. Zheng, Z. Cao, Z. Chen, Z. Li, S.M. Fei, X.H. Peng, J.F. Du, V. Vedral, NPJ Quant. Inform. 5 (2019) 39

Quantum algorithms

Quantum algorithms:

factoring, database searching, matrix inverse

large-scale, fault-tolerate universal quantum computer

Noisy intermediate-scale quantum (NISQ) processors

Variational quantum algorithms (VQAs):

classical computer + NISQ devices

Variational quantum generalized eigensolver

$$\mathcal{A}|\psi\rangle = \lambda\mathcal{B}|\psi\rangle$$

J.M. Liang, S.Q. Shen, M. Li, S.M. Fei, Quant. Inform. Processing 21 (2022) 23

J.M. Liang, S.J. Wei, S.M. Fei, Quantum gradient decent algorithms for nonequilibrium steady states and linear algebra, Sci. China Phys. Mech. & Astron. 65 (2022) 250313

Hermitian Quantum Mechanics ⇨ PT-symmetric Quantum Mechanics

M.Y. Huang, R.K. Lee, L. Zhang, S.M. Fei, J. Wu, Simulating broken PT-symmetric Hamiltonian systems by weak measurement, Phys. Rev. Lett. (2019) 080404

谢谢！

Unsupervised Tensor-Network (TN) Machine Learning Algorithm

$$|\phi_i\rangle = \prod_n |s(x_{i,n})\rangle$$

$$|s(x_{i,n})\rangle = \cos(x_{i,n}\pi/2)|0\rangle + \sin(x_{i,n}\pi/2)|1\rangle$$

$|\Psi\rangle$ optimized to minimize the negative log-likelihood

$$f = \ln|\langle\Psi|\Psi\rangle|^2 - \frac{\sum_i \ln|\langle\Psi|\phi_i\rangle|^2}{N}$$

TN: matrix product state (MPS)

$$|\Psi\rangle = \sum_{\{a\}} \prod_n \sum_{s_n=0,1} A^{[n]}_{s_n a_n, a_{n+1}} |s_n\rangle$$

{a} are known as virtual bonds of the MPS

Einstein-Podolsky-Rosen steering

$$\rho_{\mathbf{A}}^{a} = \mathrm{tr}_{\mathrm{A}}((\mathrm{M}_{\mathbf{A}}^{\mathrm{a}} \otimes \mathrm{I}_2).\rho_{\mathrm{AB}}) = \sum \mathrm{p}_\lambda \mathrm{p}(\mathrm{a}|\mathbf{A}, \lambda)\rho_\lambda$$

SDP algorithm

Semisupervised support vector machines (SVM)

Safe semisupervised SVM (S4VM)

L. Zhang, Z. Chen, S.M. Fei, Phys. Rev. A 104 (2021) 052427